Zeszyty Naukowe Uniwersytet Ekonomiczny w Krakowie

7 (943)

ISSN 1898-6447
Zesz. Nauk. UEK, 2015; 7 (943): 25–43
DOI: 10.15678/ZNUEK.2015.0943.0702

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Acceptance Control Charts

Abstract

The main goal of this article was to review and analyse basic control charts, which can be used to accept or disqualify an analysed production process. There are two kinds of acceptance control charts: Shewhart and cumulative sum (CUSUM charts). Both can be used by quality managers or financial managers in process monitoring or auditing.

Only selected process control procedures are presented in this article. Other algorithms can be found in the cited literature or developed by listed schemes. The article focuses also on the strengths and weaknesses of the proposed solutions, and provides examples of their use.

Keywords: quality control, acceptance control charts, cumulative sum control charts, statistical process control.

1. Introduction

Literature devoted to quality management or process control presents numerous tools used to monitor the course of production processes. Tools known as control charts play the major role here.

The elementary characteristic of all tools described as control charts is their ability to register, process and analyse source data which describes the process status. This analysis is aimed at detecting systematic (non-random) changes in the process, which result in the emission of a proper signal on process status.

This article reviews and analyses elementary control charts, which enable the acceptance or disqualification of the examined production process. Due to the limited size of this paper, only selected examples of control procedures will be illustrated. Other algorithms of functioning can be found in the quoted literature or elaborated according to the presented schemes. The article also looks at the strengths and weaknesses of the proposed solutions and demonstrates the ways they can be used.

2. Classification of Control Charts

Control charts can be classified in various ways, depending on different factors. Considering the number of criteria assessed when monitoring the process, univariate and multivariate charts can be distinguished. Univariate charts include charts constructed for single criteria quality assessment (single diagnostic variables) and charts using synthetic (aggregate) quality characteristics. A synthetic quality characteristic should be understood as a univariate synthetic diagnostic variable which aggregates all partial criteria of process evaluation. Multivariate charts should also be treated as including, for example, Hotelling's control charts (T^2) (see Hotelling 1947).

If one takes the theoretical conditions at work behind control charts as a criterion of division, then the group of Shewhart control charts and the group of cumulative sum control charts can be included. The former was the idea of Walter Andrew Shewhart (1891–1967), who formulated the foundations of control procedures based on the classical theory of statistical hypothesis testing in 1924, while he worked for the Western Electric Company, which manufactured telephone equipment for Bell Telephone[1]. Cumulative sum control charts are derived from the works of Abraham Wald (1902–1950) and are devoted to sequential analysis (see for example Wald 1945, 1947), which is an alternative to the classical theory of hypothesis verification.

Yet another criterion of the division of control charts that is important for this article is the possibility of process acceptance. This is related to the emission of a proper signal on the status of the monitored process. In their original construction, classical control charts (both Shewhart and cumulative sum) only allowed for a signal to be emitting, suggesting that the process is maladjusted. Such a signal resulted in the generation of feedback and the cycle of proper activities aimed at restoring the desired status. The issue of generating the signal that would suggest that the process is adjusted was less important. Over time, there appeared a need for process acceptance, which entailed the relevant modification of classical control charts.

[1] The concept of classical procedures of statistical hypothesis testing should be understood as procedures proposed by Jerzy Spława-Nayman and Egon Pearson.

Such modifications concerned both Shewhart control charts (see Iwasiewicz 1985, pp. 57–86, 159–163; 1999, pp. 239–242; 2001, pp. 35–48; see also the standard PN-ISO 7966: 2001), and cumulative sum control charts (see Major 1997, pp. 47–54; Iwasiewicz 2008–2009; 2011, pp. 213–245; Major 2015, pp. 223–238).

New charts enable the disqualification or acceptance of the process being examined. The possibility of process acceptance becomes very important for the broadly understood process of company management; a control chart of this type is an important tool in the hands of a quality manager or financial manager.

3. Process Analysis Based on Shewhart Control Charts

These charts belong to the earliest and probably best known tools for the statistical control of production processes. They are simple overview diagrammes which make it possible to determine quickly whether a given process has been destabilised. They owe their popularity and recognition among engineers, controllers and academics mainly to their simplicity and the lack of the need to use advanced computational techniques. This was particularly important in the 1920s when there were no computers or calculators and a control chart could be made with no more than a sheet of paper and a pencil. Today, in the era of the Internet and advanced computerisation, the graphic form of Shewhart procedures is reduced to a role auxiliary to numeric form. Currently, Shewhart control charts are seen from the perspective of mathematical statistics, or, to be more precise, the theory of significance tests. Such a chart can be defined as a sequence of relevant significance tests. As a result, just as with significance testing, in Shewhart control charts it is assumed that a Type I (α) error can be committed – it is referred to as the risk of excessive process adjustment. In the classical control chart one does not define *explicite* the likelihood of making Type II (β) error, which renders it impossible to make a decision on accepting the monitored process. However, it can only be stated that there are no reasons for rejecting the zero hypothesis that the process is adjusted. Obviously, each standard Shewhart control chart can be properly modified so that it would be possible to take a decision to either accept or disqualify the process.

In order to illustrate these possibilities, we should focus on the case when during process x – the mean chart is used and it functions with the bilateral control scheme. It can be assumed that the values of diagnostic variable X with a distribution similar to normal and with constant and known standard deviation σ are observed.

In this case, the second of the distribution parameters – the expected value – is the subject of verification. In the standard control procedure (with the bilateral control scheme), in the subsequent observation points of the process $t = 1, 2, 3, \ldots$, a null hypothesis of the following form is used:

$$H_0: \mu_t = \mu_0, \tag{1}$$

in contrast to the alternative hypothesis:

$$H_1: \mu_t \neq \mu_0, \tag{2}$$

where μ_t is the expected value in period t, while $\mu_0 = x_0$ is the target value (nominal, most desired).

The observed characteristic from the sample is the arithmetic mean of the following form:

$$\bar{x}_t = \frac{\sum_{i=1}^{n} x_{ti}}{n}. \tag{3}$$

It is assumed that the size of the examined sample taken for examination in each period t is constant and amounts to n. The values of characteristics from the sample are placed on the overview diagramme and compared to the levels of control limits determined according to the following formulas:

$$\bar{x}_u = x_0 + u_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \tag{4}$$

$$\bar{x}_l = x_0 - u_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \tag{5}$$

$$\bar{x}_0 = x_0, \tag{6}$$

where $\bar{x}_u$ is the upper control limit, $\bar{x}_l$ is the lower control limit, $\bar{x}_0$ is the centre line, while $u_{\frac{\alpha}{2}}$ is the quantile of standard normal random variable $U \sim N(0, 1)$ which fulfills the following condition:

$$\mathrm{P}(U > u_{\frac{\alpha}{2}}) = \frac{\alpha}{2}. \tag{7}$$

A point signal on process maladjustment[2] is emitted – with the probability of a α error – then the following condition is met:

$$\bar{x}_t < \bar{x}_l \text{ or } \bar{x}_t > \bar{x}_u. \tag{8}$$

[2] The procedure described here refers to the simplest situation when the point signal of process maladjustment is generated. The emission of serial signals on maladjustment and warning signals or signals from combined samples is also possible (more – see Iwasiewicz 1985, pp. 108–113; PN-ISO 8258+AC1: 1996).

In other cases, it is said that there are no grounds for rejecting the null hypothesis, which is not equivalent to an unambiguous process acceptance.

The above chart may be modified so that a null hypothesis with the proper error risk (β) could be assumed.

Then, an alternative hypothesis is also modified (2). It can be substituted by two alternative hypotheses in the following form (after Iwasiewicz 2001, p. 240):

$$H_{-1}\colon \mu_t = x_0 - \Delta\mu = \mu_{-1}, \tag{9}$$

$$H_1\colon \mu_t = x_0 + \Delta\mu = \mu_1, \tag{10}$$

whereas $\Delta\mu > 0$, H_{-1} and H_1 stand for the left-tailed and right-tailed hypotheses.

With the values μ_{-1} and μ_1, the location of control lines ($\bar{x}_l$ and $\bar{x}_u$) is determined at such a level that:

$$P(\bar{x}_t < \bar{x}_l \mid \mu_t = x_0) = \frac{\alpha}{2} \text{ and } P(\bar{x}_t > \bar{x}_l \mid \mu_t = \mu_{-1}) = \frac{\beta}{2}, \tag{11}$$

and simultaneously:

$$P(\bar{x}_t > \bar{x}_u \mid \mu_t = x_0) = \frac{\alpha}{2} \text{ and } P(\bar{x}_t < \bar{x}_u \mid \mu_t = \mu_1) = \frac{\beta}{2}. \tag{12}$$

The lower and upper control limits are determined according to the same formulas (4) and (5), but with the assumption that the minimum size of the sample ($n = n^*$) which guarantees that conditions (11) and (12) are met should amount to at least[3]:

$$n^* = \left(\frac{\sigma\left(u_{\frac{\alpha}{2}} + u_{\frac{\beta}{2}}\right)}{\Delta\mu}\right)^2. \tag{13}$$

Formula (13) was created as a result of a solution to n system of equations that describes the location of line $\bar{x}_u$ (after Iwasiewicz 1999, p. 240; 2001, pp. 40–41):

$$\left.\begin{aligned} \bar{x}_u &= x_0 + u_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \\ \bar{x}_u &= \mu_1 - u_{\frac{\beta}{2}} \frac{\sigma}{\sqrt{n}} \end{aligned}\right\}, \tag{14}$$

where:

$$\mu_1 = x_0 + \Delta\mu. \tag{15}$$

The decision rules are as follows.

[3] See (Iwasiewicz 1999, p. 240). Cf. (Iwasiewicz 1985, p. 70; PN-ISO 7966: 2001, p. 9).

If:

$$\bar{x}_l \leq \bar{x}_t \leq \bar{x}_u, \tag{16}$$

then the course of the process is accepted and the likelihood of a random event is such that the process maladjustment reaches or exceeds $\Delta\mu$, but does not exceed β.

However, if:

$$\bar{x}_t < \bar{x}_l \text{ or } \bar{x}_t > \bar{x}_u, \tag{17}$$

it is assumed that the maladjustment of the monitored process has reached $\Delta\mu$ value and the probability of a random event of the actual adjustment of the process ($\mu = x_0$) does not exceed α.

In the case of the one-sided control schemes in formulas (13) and (14), $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ should be substituted by α and β. Decision-making will also look different. Then, the process should be considered adjusted if: $\bar{x}_t \geq \bar{x}_l$ for the left-sided scheme and $\bar{x}_t \leq \bar{x}_u$ for the right-sided scheme. The signal on process maladjustment is sent when: $\bar{x}_t < \bar{x}_l$ for the left-sided scheme and when: $\bar{x}_t > \bar{x}_u$ for the right-sided one.

The relevant nomograms or numerical procedures used when designing process acceptance charts can be found in the standards (see PN-ISO 7966: 2001). Along with the determination of the minimum size of the sample, they make it possible to determine other chart parameters like μ_{-1} and μ_0 in combination with the highest permitted defectiveness level p_0 and the lowest disqualifying defectiveness p_1, as well as tolerance range boundaries. Values μ_{-1} and μ_0 may also be determined based on substantive conditions and knowledge of the technological process.

4. Analysis of the Process Based on Classical Sequential Procedures

The procedures enable both the acceptance and the disqualification of the process being examined. They are based on sequential probability ratio tests and, more broadly, on the sequential analysis mentioned above.

The theory of sequential analysis involves random sampling of single or small sets of statistical populations and the determination each time of whether the information resources obtained to that point allow for a specific decision to be made. With the formulated null hypothesis H_0 and alternative hypothesis H_1, such decisions may concern:

– the adoption of H_0 hypothesis,

– the rejection of H_0 hypothesis and the adoption of H_1 hypothesis,

– the postponement of the decision until the next unit (sample n) is taken for sample m.

In this view, sample *m* is the sum of all samples *n* collected in subsequent steps *k* – that is:

$$m = n_1 + n_2 + \ldots + n_k. \tag{18}$$

If in each *k*-th sampling interval the sample size is one, then the size of the whole cumulative sample will be *k* ($m = k$).

As previously, the considerations were reduced to the case when the diagnostic variable *X* has the normal distribution of the expected value μ and known standard deviation σ ($X \sim N(\mu,\sigma)$). Similarly as before, it was assumed that the control concerned the expected value μ. Due to the limited size of the paper, the analysis was narrowed down this time to the situation when the tolerance interval was right-hand limited[4].

The observed characteristic from the sample is the sum of the realisation of the tested diagnostic variable in subsequent steps of the sequential test. With the assumption that in each step of the sequential analysis there is only one realisation of the random variable available, the characteristic from the sample will take the following form:

$$z_n = \sum_{i=1}^{n} x_i, \; n = 1, 2, \ldots, \tag{19}$$

where x_i is the realisation of diagnostic variable *X* in subsequent steps of the sequential procedure.

The verified null hypothesis takes the form (1) while the alternative one has the form (10). Between the values μ_0 and μ_1 there is relation $\mu_0 < \mu_1$.

The examined stochastic process is said to be adjusted and the product to comply with quality requirements when the following dependency is fulfilled: $\mu \le \mu_0$. When $\mu \ge \mu_1$, it is said that the process is not adjusted and the product does not fulfill quality requirements. When $\mu_0 < \mu < \mu_1$, one of the two decisions should be postponed until more information is obtained.

One of the two above decisions is made with the assumed risk of error α and β, where:

α is the probability that hypothesis H_1 will be adopted when H_0 is true, and

β is the probability that hypothesis H_0 will be adopted when H_1 is true.

During the verification, sequential testing in the following form is used (Iwasiewicz, Paszek & Steczkowski 1988, p. 46):

[4] An extensive description of other cases can be found, for example, in the paper (Iwasiewicz, Paszek & Steczkowski 1988).

$$\frac{\beta}{1-\alpha} < \frac{\prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_i-\mu_1}{\sigma}\right)^2}}{\prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_i-\mu_0}{\sigma}\right)^2}} < \frac{1-\beta}{\alpha}, \quad (20)$$

which is the quotient of two reliability functions formulated for cases when $\mu = \mu_1$ and $\mu = \mu_0$. Parameters $1 - \alpha$ and $1 - \beta$ complement α and β, and they respectively stand for: the probability of adopting H_0 hypothesis when it is true $(1 - \alpha)$ and the probability of adopting H_1 hypothesis when it is true $(1 - \beta)$.

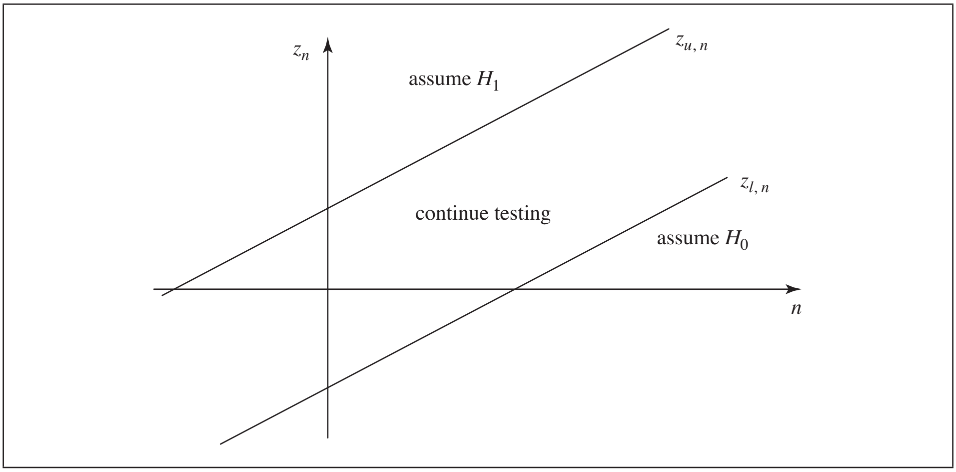

Fig. 1. Overview Diagramme Used in the Sequential Procedure of Testing Hypothesis H_0: $\mu = \mu_0$, against hypothesis H_1: $\mu = \mu_1$

Source: the author's study.

When transforming formula (20) in relation to statistics z_n in the form of (19), we obtain:

$$\frac{\sigma^2}{\mu_1-\mu_0}\ln\frac{\beta}{1-\alpha} + \frac{\mu_1+\mu_0}{2}n < \sum_{i=1}^{n} x_i < \frac{\sigma^2}{\mu_1-\mu_0}\ln\frac{1-\beta}{\alpha} + \frac{\mu_1+\mu_0}{2}n. \quad (21)$$

Inequality (21) can also be written in the following form:

$$z_{l,n} = a + cn < z_n < z_{u,n} = b + cn, \quad (22)$$

where:

$$a = \frac{\sigma^2}{\mu_1-\mu_0}\ln\frac{\beta}{1-\alpha}, \quad (23)$$

$$b = \frac{\sigma^2}{\mu_1-\mu_0}\ln\frac{1-\beta}{\alpha}, \quad (24)$$

$$c = \frac{\mu_1 + \mu_0}{2}. \quad (25)$$

The rules of the procedure are as follows:

If in any step of the test the following inequality is fulfilled: $z_n < z_{l,n}$, then H_0 hypothesis is assumed (and the examined process is adjusted) as is the probability of H_1 hypothesis being true and not exceeding β. If the following inequality is true: $z_n > z_{u,n}$, then H_1 hypothesis is assumed (and the examined process is maladjusted) as is the probability of H_0 hypothesis being true and not exceeding α. Whereas: if $z_{l,n} \le z_n \le z_{u,n}$, there are no grounds for adopting any of the hypotheses and testing should be continued with the size of the sample increased by one and parameters $z_{l,n}$, $z_{u,n}$ and z_n calculated again. The calculated values z_n can also be analysed in a graphic form with the use of the diagramme presented in Fig. 1.

5. Classical Procedure of Cumulative Sums

The rule of the functioning of these control charts was discussed based on the assumption from the previous paragraph that the expected value of the diagnostic variable with normal distribution, with fixed and known standard deviation, was verified. Furthermore, the assumption of right-sided limit to the tolerance interval remains.

When constructing the procedure of cumulative sums, it is assumed that the risk of Type II error β = 0. As a result, the region of test continuation is combined with the area where hypothesis H_0 was assumed, so the number of possible decisions is reduced to two. The whole technique behind the procedure is based on the assumption that the procedure of cumulative sums is a classical backward sequential procedure. This forces the drafting of an auxiliary coordinate system in each point ending a given sequence, rotated by 180° in relation to the original position. An example of such a graph is presented in Fig. 2.

In each n-th step of the procedure it is verified if the sequence $z_n = \sum_{i=1}^{n} x_i$ observed to that point is sufficient for adopting hypothesis H_1 (determination that the process is maladjusted).

For the greater convenience of this method, a mask is created and moved on the graph with the increase in the length of the examined sequence. An example of such a mask is presented in Fig. 3.

In order to construct a mask, several parameters must be known, including edges d and h and angle φ, which is an inclination angle of an active edge of the mask to the abscissa of the system of coordinates turned by 180°.

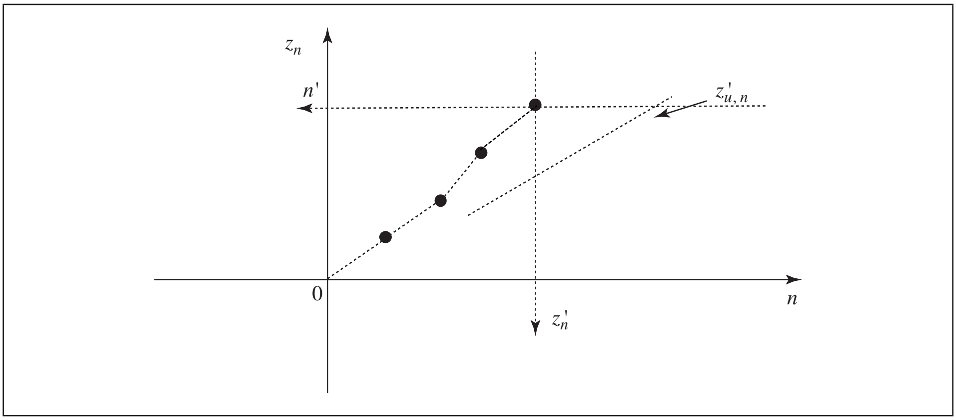

Fig. 2. Dependency between the Classical Procedure and the Procedure of Cumulative Sums

Source: the author's study.

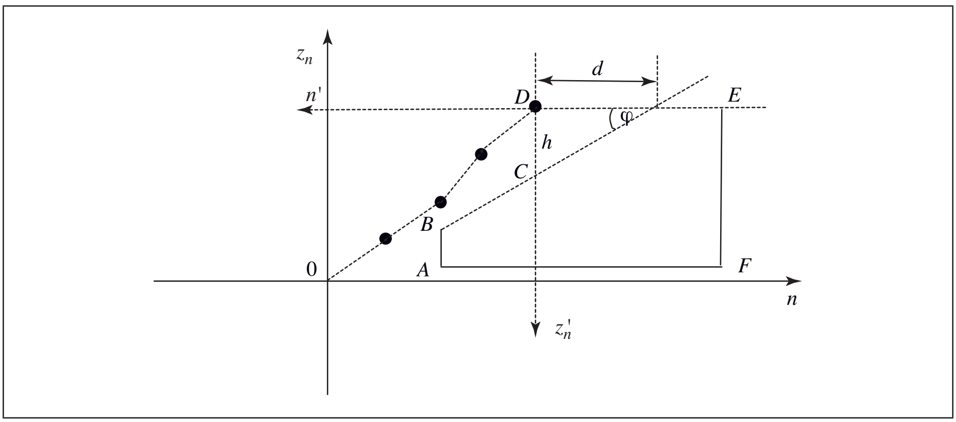

Fig. 3. Classical Mask in the Control Scheme with the Right-sided Limit to the Tolerance Interval

Source: the author's study.

The value of parameter d is determined by calculating the zero of the equation of control line z_g with the assumption that $\beta = 0$. Considering the fact that the parameters of the mask are determined in the coordinate system turned by 180°, it must be written:

$$d = -n = -\frac{2\sigma^2 \ln \alpha}{\mu_1^2 - \mu_0^2} > 0. \qquad (26)$$

The second of the mask parameters, the inclination of the active edge of the mask, may be determined using this formula:

$$\varphi = arctgc = arctg\frac{\mu_0 + \mu_1}{2}. \tag{27}$$

The last of the parameters of h mask is determined with this formula:

$$h = dtg\varphi = dc. \tag{28}$$

The active control edge of the mask (segment BC in Fig. 3) and point D covering the last of the points of the observed sequence are most important in the mask. Another important rule is that the upper edge of mask DE should remain parallel to the abscissa n.

The rules of the procedure are as follows:

– if at least one point in sequence $z_0 \ldots z_n$ is below the active control edge (line) BC, then H_1 hypothesis should be adopted (the process is maladjusted) with the error risk not exceeding α;

– if none of the sequence points lies below edge BC, then one should go to the next stage of the test and increase the cumulative sample by the next value $x_{i.}$

In many cases, the use of the mask is quite problematic, especially when its parameters are expressed by high values. This inconvenience can be removed by substituting a classical graphic algorithm with a numerical algorithm. The rules are quite similar to the rules in Shewhart procedures while parameters found in the formulas are then designated similarly to the mask parameters.

If it is assumed that $\beta = 0$, the statistics z_n can be replaced by z_n^* statistics defined by the following formula:

$$z_n^* = \sum_{i=1}^{s} (x_{ti} - c), \tag{29}$$

where $c = tg\varphi$, t is the current index, i is the operational index, and s is the highest value of operational index in a given moment ($s = 1, 2, 3, \ldots$).

Double index it in formula (29) suggests the tests are conducted in two stages. Index t has functioned for the entire duration of tests while index i appears when the following dependency is met: $x_t - c > 0$. The fulfillment of the above inequality is a necessary condition for starting the calculation of the value of statistic (29). The calculation of characteristic (29) is continued until one of the conditions below is met:

$$1^0 \;\; z_n^* \geq z_u, \text{ where } z_u = dc = h, \tag{30}$$

$$2^0 \;\; z_n^* \leq 0. \tag{31}$$

If condition 1^0 (30) is met, the test is concluded and alternative hypothesis H_1 is assumed with the risk of error not greater than *a,* whereas if condition 2^0 (31) is met, the calculation of the values of characteristic z_n^* is discontinued and value 0 is attributed to index *i*.

6. Modified Procedure of Cumulative Sums

The algorithms presented in the preceding point both in graphic and numeric form can be modified in such a way that the acceptance of the examined process would be possible. For this purpose, one should avoid combining process acceptance areas with the area of test continuation (see Fig. 1). In practice, this means that $\beta > 0$ and the mask will slightly change its structure. An example of such a mask is shown in Fig. 4.

The mask now has two active edges (control lines), rather than one. If the lower line of the mask is exceeded in minus, this suggests the need to adopt hypothesis H_1 (the process is maladjusted) with the error risk not greater than α. If the upper line of the mask is exceeded in plus, hypothesis H_0 is assumed (the process is adjusted) with the error risk not greater than β. If points of the cumulative sequence are arranged in the corridor between the mask edges, then tests are continued with the sample increased by subsequent value x_i. Mask parameters are determined according to rules similar to those for a classical procedure of cumulative sums, yet with one difference – the value of the β parameter is greater than zero. Currently, in order to build a mask, one needs (see Fig. 4) parameters d_0, d_1, h_0, h_1 and $c = tg\varphi$.

Parameter d_0 is obtained as a result of the transformation of the equation of the lower control limit $z_{l,n}$ of the classical sequential procedure (see Fig. 1). The transformation involves the determination of the zero n_0 of function $z_{l,n}$. Remembering that the mask functions in the coordinate system turned by 180°, the following can be written:

$$d_0 = -n_0 = \frac{2\sigma^2 \ln \frac{\beta}{1-\alpha}}{\mu_1^2 - \mu_0^2} < 0. \tag{32}$$

The equation of parameter d_1 is set similarly, but this time by transforming the upper control limit $z_{u,n}$. In this way, we obtain the following formula:

$$d_1 = \frac{2\sigma^2 \ln \frac{1-\beta}{\alpha}}{\mu_1^2 - \mu_0^2} > 0. \tag{33}$$

The value of parameter φ is determined according to formula (27), while parameters h_0 and h_1 are taken from the following formulas:

$$h_0 = d_0\, tg\varphi = d_0\, c = \frac{\sigma^2 \ln \frac{\beta}{1-\alpha}}{\mu_1 - \mu_0} < 0, \tag{34}$$

$$h_1 = d_1\, tg\varphi = d_1\, c = \frac{\sigma^2 \ln \frac{1-\beta}{\alpha}}{\mu_1 - \mu_0} > 0. \tag{35}$$

The values of parameters h_0, h_1 and c are necessary during the construction of the process status numerical algorithm. This algorithm then proceeds like a standard procedure of cumulative sums. The difference is that one can distinguish two different process sequences: one leads to process acceptance (adoption of hypothesis H_0) and the other to its disqualification (adoption of hypothesis H_1). In both cases, the cumulative value of statistic z_n^* in the form (29) is tested. Just like before, index t in formula (29) functions throughout the entire duration of tests while the moment of starting index i depends on the kind of point sequence used.

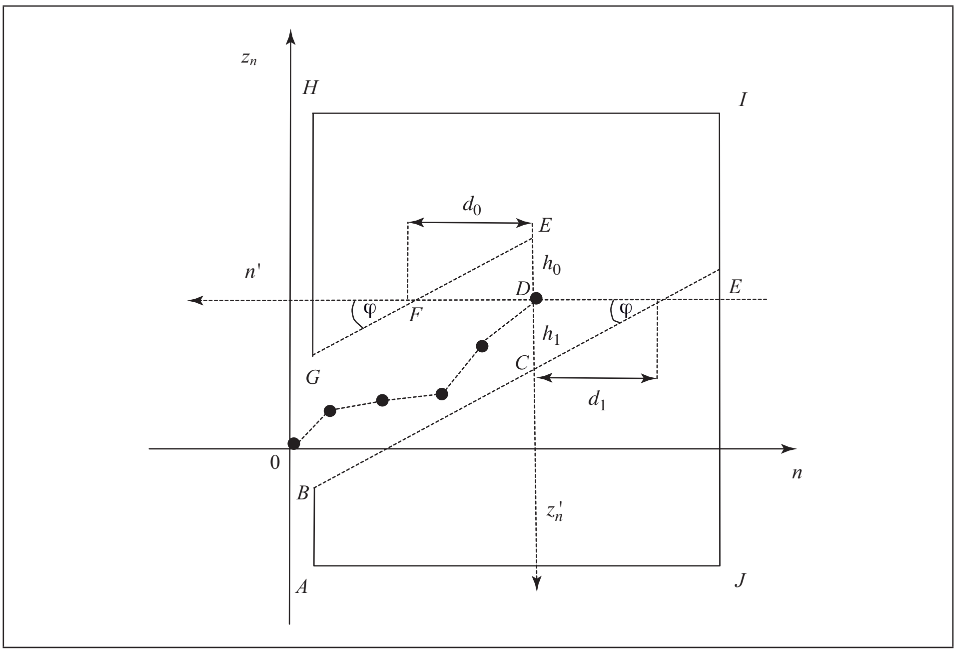

Fig. 4. A Modified Mask in the Control Scheme with the Right-hand Tolerance Interval Limit

Source: the author’s study.

The analysis begins with the subtraction sign $x_t - c$. When $x_t - c < 0$, counter i is launched and a sequence begins which may lead to the adoption of hypothesis H_0. The fact that the above inequality is met is a necessary condition for starting the calculation of statistic value (29). The calculation of this statistic is continued as long as one of the conditions below is met:

$$1^A \quad z_n^* \leq z_l^*, \text{where } z_l^* = d_0 c = h_0, \tag{36}$$

$$2^A \quad z_n^* \geq 0. \tag{37}$$

When condition 1^A (36) is met, the test ends with the adoption of null hypothesis H_0 with error risk not greater than β; when condition 2^A (37) is fulfilled, the calculation of the value of characteristic z_n^* is interrupted and one returns to tracking the subtraction sign $x_t - c$. The fulfillment of conditions 1^A (36) or 1^B (37) also entails the need to bring index i ($i = 0$) to zero.

Index i may also be started when $x_t - c > 0$. Then, it is the beginning of the second sequence which can lead to the adoption of alternative hypothesis H_1 and the beginning of accumulation according to formula (29). Such accumulation lasts until one of the conditions below is met:

$$1^B \quad z_n^* \geq z_u^*, \text{ where } z_u^* = d_1 c = h_1, \tag{38}$$

$$2^B \quad z_n^* \leq 0. \tag{39}$$

The fulfillment of condition 1^B (38) results in the adoption of alternative hypothesis H_1 with a risk error not greater than α. The fulfillment of the second condition 2^B (39) means the immediate need to stop the accumulation and return to the tracking of the subtraction sign $x_t - c$. Just like in the previous case, the fulfillment of condition 1^B or 2^B results in the bringing of index i ($i = 0$) to zero.

7. Examples of Control Chart Use

Example 1

Preservative is automatically added to fruit juice during the bottling process. This preservative is not altogether healthy for consumers. An optimum quantity of the preservative which does not pose a threat to the health of the consumer and guarantees the freshness of the product is x_0 = 140 mg/l plus minus 20 mg/l. The precision of the dispenser is constant at σ = 10 mg. If it is assumed that the content of the substance in 1 litre of the beverage is a random variable with normal probability distribution, a Shewhart control chart that will allow for the acceptance or disqualification of the process of dispensing the preservative should be constructed. When constructing the chart, it should be assumed that α = β = 0.1.

As the analysed diagnostic variable is the nominal best of the examined process, the following hypotheses are subject to verification:

H_0: $\mu_t = x_0 = 140$ mg/l,

H_{-1}: $\mu_t = x_0 - \Delta\mu = \mu_{-1} = 120$ mg/l,

H_1: $\mu_t = x_0 + \Delta\mu = \mu_1 = 160$ mg/l.

The value of difference $\Delta\mu = 20$ was adopted arbitrarily in this case and was linked to the length of the tolerance interval. In some cases, the difference $\Delta\mu$ may be reduced to the level of, e.g., 10 or 5 mg/l in order to improve the selectiveness of the designed procedure.

With the use of statistical charts of standard normal distribution, quantiles: $u_\alpha = u_\beta = 1.645$ can be read. The minimum size of the sample $n^* = 2.7 \approx 3$ is calculated based on formula (13). The location of adjustment limits will be, respectively, formulas (4) and (5):

$$\bar{x}_u = 140 + 1.645\frac{10}{\sqrt{3}} \approx 149.5,$$

$$\bar{x}_l = 140 - 1.645\frac{10}{\sqrt{3}} \approx 130.5.$$

Thus, if the average quantity of the substance calculated based on the randomly chosen sample of at least 3 juices fulfills inequality (16) – that is, $130.5 \leq \bar{x}_t \leq 149.5$ – then hypothesis H_0 should be assumed and the probability that in fact hypothesis H_{-1} is true or hypothesis H_1 does not exceed $\beta = 0.1$. If the computed mean of the sample lies beyond the area limited by control lines $\bar{x}_l \approx 130.5$ and $\bar{x}_u \approx 149.5$, then it should be stated that the examined process needs adjustment and the likelihood of it being redundant does not exceed $a = 0.1$.

Example 2

The settlement period for issued invoices was monitored. For this purpose, invoices were chosen randomly at the end of each month and the period that lapsed between the issuance of the invoice and the receipt of payment was observed. The necessary condition for financial liquidity is the average time of 7 days for the invoice to be paid. What can be said about the payment process if the values shown below were noted for 10 randomly chosen invoices?

Table 1. Period of Invoice Settlement

i	1	2	3	4	5	6	7	8	9	10
n	1	2	3	4	5	6	7	8	9	10
x_i	6	7	5	4	9	7	8	6	5	6

Source: conventional data.

During the analysis of the above process, it can be assumed that: $\alpha = \beta = 0.05$ and the period of invoice settlement has a normal distribution with constant variance of 4.

Because the diagnostic variable of the period of invoice settlement is the smaller-the-better, the null hypothesis and the alternative hypothesis will take the following form:

H_0: $\mu = \mu_0 = 7$,

H_1: $\mu = \mu_1 = 8$.

With the use of formulas (23), (24) and (25), the values of parameters a, b and c necessary for applying the classical sequential procedure can be set.

$$a = \frac{2^2}{8-7}\ln\frac{0.05}{0.95} = -11.78,\ b = \frac{2^2}{8-7}\ln\frac{0.95}{0.05} = 11.78,\ c = \frac{8+7}{2} = 7.5.$$

Thus, one obtains:

$z_{l,n} = -11.78 + 7.5n$,

$z_{u,n} = 11.78 + 7.5n$.

The results of the analysis of empirical data are shown in Table 2.

Table 2. Analysis of the Invoice Settlement Process with the Use of the Classical Sequential Procedure

i	n	x_i	z_n	$z_{l,n}$	$z_{u,n}$	Decision
1	1	6	6	–4.28	19.28	Continue testing
2	2	7	13	3.22	26.78	Continue testing
3	3	5	18	10.72	34.28	Continue testing
4	4	4	22	18.22	41.78	Continue testing
5	5	9	31	25.72	49.28	Continue testing
6	6	7	38	33.22	56.78	Continue testing
7	7	8	46	40.72	64.28	Continue testing
8	8	6	52	48.22	71.78	Continue testing
9	9	5	57	55.72	79.28	Continue testing
10	10	6	63	63.22	86.78	Assume H_0

Source: the author's study.

The above analysis shows that in the tenth step of the sequential process, with the size of the cumulative sample $n = 10$, the null hypothesis H_0 should be assumed and the payment process should be considered adjusted. The probability of this evaluation being false does not exceed $\beta = 0.05$.

A similar analysis can be conducted based on the modified procedure of cumulative sums. Then the values of parameters: d_0, d_1 and h_0, h_1 must be additionally computed (see formulas (32) to (35)).

The parameters are:

$$d_0 = \frac{2 \cdot 2^2 \ln \frac{0.05}{0.95}}{8^2 - 7^2} = -1.57, \; d_1 = \frac{2 \cdot 2^2 \ln \frac{0.95}{0.05}}{8^2 - 7^2} = 1.57,$$

$$h_0 = d_0 c = z_d^* = -11.78, \; h_1 = d_1 c = z_g^* = 11.78.$$

Having these parameters, re-analysis of the data can be started as shown in Table 3.

Table 3. Analysis of the Process of Invoice Settlement Using the Modified Procedure of Cumulative Sums

T	i	x_{ti}	n	$z_i^* = x_{ti} - c$	z_n^*	Comments
1	1	6	1	–1.5	–1.5	x_t < c (start accumulation, i = 1)
2	2	7	2	–0.5	–2	accumulate ($i = i + 1$)
3	3	5	3	–2.5	–4.5	accumulate ($i = i + 1$)
4	4	4	4	–3.5	–8	accumulate ($i = i + 1$)
5	5	9	5	1.5	–6.5	accumulate ($i = i + 1$)
6	6	7	6	–0.5	–7	accumulate ($i = i + 1$)
7	7	8	7	0.5	–6.5	accumulate ($i = i + 1$)
8	8	6	8	–1.5	–8	accumulate ($i = i + 1$)
9	9	5	9	–2.5	–10.5	accumulate ($i = i + 1$)
10	10	6	10	–1.5	–12	$z_n^* < z_l^*$ (assume H_0)

Source: the author’s calculations.

It is clear that the modified procedure of cumulative sums also needed ten steps to make a decision on the adoption of the null hypothesis. The accumulation process began in period $t = 1$ and lasted uninterrupted until period $t = 10$.

8. Conclusion

The behaviour algorithms presented in points 4 through 6 have been deliberately narrowed down to the case in which a diagnostic variable has normal distribution, with known standard deviation and the tolerance interval limited on the right-hand side. Similarly, one can also build behaviour schemes for cases when the tolerance interval is limited on the left-hand side and on both sides. The modification of cumulative sum control charts can also be performed for cumulative sum control charts used to assess the quality due to discrete diagnostic variables. Examples of such solutions can be found in (Iwasiewicz 2008–2009; 2011, and Major 2015).

An elementary advantage of the use of modified cumulative sum control charts is that, just like Shewhart control charts, they do not require preliminary determination of the size of the sample necessary for assuming one of the verified hypotheses. In sequential procedures and cumulative sum procedures based on them, the size of the sample is a parameter determined only at the time of accepting or disqualifying the process. Thanks to that parameter, there is no option to overestimate the sample size and generate excessive control costs. For this reason, sequential methods seem most efficient and cost effective when compared to the other process evaluation methods presented in this article.

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Karty kontrolne z możliwością akceptacji procesu

(Streszczenie)

Celem artykułu jest przegląd oraz analiza podstawowych kart kontrolnych umożliwiających zarówno akceptację, jak i dyskwalifikację badanego procesu produkcyjnego. Istnieją dwa rodzaje kart kontrolnych umożliwiających akceptację procesu: karty kontrolne Shewharta oraz karty kontrolne sum skumulowanych. Obie mogą być wykorzystywane przez menedżerów jakości lub menedżerów finansowych w trakcie monitorowania lub audytu procesu.

W artykule zostały zilustrowane tylko wybrane przykładowe procedury kontrolne. Pozostałe algorytmy można znaleźć w cytowanej literaturze lub opracować według zamieszczonych schematów. W artykule zwrócono również uwagę na mocne i słabe strony proponowanych rozwiązań oraz podano przykłady ich zastosowania.

Słowa kluczowe: kontrola jakości, karty akceptacji procesu, karty kontrolne sum skumulowanych, statystyczne sterowanie procesem.