dr Agnieszka Rygiel
Katedra Matematyki
Uniwersytet Ekonomiczny w Krakowie

SUPER-REPLICATION OF EUROPEAN OPTIONS WITH CONVEX PAYOFF UNDER PROPORTIONAL TRANSACTION COSTS

1. Introduction

In the paper we consider the problem of super-replication of the European options in a discrete time market model with transaction costs and volatility uncertainty. The problem of pricing and hedging of contingent claims under proportional transaction costs was studied in a number of papers [Stettner 1997, Kociński 2004, Roux, Tokarz and Zastawniak 2008]. Option pricing is closely related to the aspect of arbitrage under transaction costs studied in [Guasoni, Lépinette and Rásonyi 2012, Guasoni, Lépinette and Schachermayer 2008, Jouini and Kallal 1995, Schachermayer 2003]. A classical probabilistic model of a financial market consists of a measurable space $(\Omega, \mathcal{F})$ and a probability measure $\mathbb{P}$ determining the distribution of stock prices. In contrast, we study a situation where the distribution is not assumed to be (completely) known a priori. Our sole assumption on the stock price dynamic is that the absolute value of the log-returns is bounded from below and above. The problem of super-replication in such a discrete-time model, with more general nonlinear transaction costs, is studied in [Bank, Dolinsky and Gökay 2016]. The result, including the special case of a convex payoff profile, seems to be nontrivial to apply to the real market from the calculation point of view. The paper of Roux, Tokarz and Zastawniak [2008] develops the pricing and hedging algorithm in the discrete setting under proportional transaction costs, which can be rapidly implemented on a computer, but does not allow for general Ω which does not have to be finite. The aim of this paper is to show that pricing of convex European options can be reduced to study pricing of suitable Binomial model following some arguments of Bank, Dolinsky and Gökay [2016] as well as Roux, Tokarz and Zastawniak [2008].

2. Model description

Let us consider a discrete time market model with a time horizon $N \in \mathbb{N}$ with a riskless savings account $B_n = 1$, $n = 0, \ldots, N$ and a risky stock $S_n > 0$, $n = 0, \ldots, N$. Let

$$X_n := \ln\left(\frac{S_n}{S_{n-1}}\right) \qquad (1)$$

be the log-return for period $n = 1, \ldots, N$ such that

$$\underline{\sigma} \leq |X_n| \leq \overline{\sigma}, \; n = 1, \ldots, N \qquad (2)$$

for some constants $0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$. Hence

$$S_n = s_0 exp\left(\sum_{m=1}^{n} X_m\right),\ n = 0,\dots,N. \tag{3}$$

Let

$$\Omega = \Omega_{\underline{\sigma},\bar{\sigma}} := \left\{\omega = (x_1,\dots,x_N) \in \mathbb{R}^N :\ \underline{\sigma} \le |x_n| \le \bar{\sigma},\ \ n = 1,\dots,N\right\} \tag{4}$$

be the path space with the canonical process

$$X_k(\omega) := x_k \ \ for\ \ \omega = (x_1,\dots,x_N) \in \Omega \tag{5}$$

and the canonical filtration

$$\mathcal{F}_n := \sigma(X_1,\dots,X_n),\ n = 0,\dots,N. \tag{6}$$

For every $n = 0, \dots, N$ the cost of buying one share of the stock at time n is $S_n(1+\lambda)$, where $\lambda \in [0,\infty)$ and the amount received for selling one share at time n is $S_n(1-\mu)$ with $\mu \in [0,1)$. A trading strategy is a pair (η,θ) of predictable processes η_n, θ_n representing positions in cash and stock respectively, at $n = 0, \dots, N$. We define the time n liquidation value ϑ_n of a portfolio (α,β) of cash and stock as

$$\vartheta_n(\alpha,\beta) = \alpha + \beta^+(1-\mu)S_n - \beta^-(1+\lambda)S_n. \tag{7}$$

Definition 1 *We call a pair (η,θ) of predictable processes η_n, θ_n a self-financing strategy, if*

$$\vartheta_n(\eta_n - \eta_{n+1}, \theta_n - \theta_{n+1}) \ge 0 \tag{8}$$

for each $n = 0, \dots, N-1$, with $\theta_0 = 0$.

We shall denote the class of self-financing strategies starting with initial capital η_0 by $\Phi(\eta_0)$.

Definition 2 *The super-replication price of a European option $F\colon \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ is defined as*

$$\begin{aligned}\pi_{\underline{\sigma},\bar{\sigma}}(F) = inf\{&\eta_0 \in \mathbb{R} \,|\, \exists(\eta,\theta) \in \Phi(\eta_0):\\ &\vartheta_N\left(\eta_N(\omega) - F(S(\omega)), \theta_N(\omega)\right) \ge 0\ \ \forall \omega \in \Omega_{\underline{\sigma},\bar{\sigma}}\}.\end{aligned} \tag{9}$$

We are interested in determining the minimal initial capital which allows, by using self-financing strategy, to end up holding a solvent portfolio $\big(\eta_N(\omega) - F\big(S(\omega)\big), \theta_N(\omega)\big)$, by delivering the payoff $F(S)$ at time N. We emphasize that we do not define super-replication prices in an almost sure-sense as in classical approach.

3. Main result

We consider the special case of a convex payoff profile and show that the super-replication price coincides with the one computed in the Binomial model where the volatility always takes its maximal values. This result allows us to continue our investigation in the finite path space and consequently to use an algorithmic approach to optimal super-replication of European options considered in [Roux, Tokarz and Zastawniak 2008].

Theorem 3 *Suppose that the payoff function* $F\colon \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ *is convex. Then the super-replication price of any European option with payoff* F *is given by*

$$\pi_{\underline{\sigma},\bar{\sigma}}(F) = \bar{\pi}(F)$$

where $\overline{\pi}(F) = \pi_{\overline{\sigma},\overline{\sigma}}(F)$ *denotes the super-replication price of* $F(S)$ *in the Binomial model with volatility* $\overline{\sigma}$ *and transaction costs* λ, μ.

A similar result for markets with friction is shown in [Bank, Dolinsky and Gökay 2016]. Note that in [Bank, Dolinsky and Gökay 2016] the mark-to-market value rather than the liquidation value is considered. Moreover, our approach is based on a different definition of the self-financing strategy.

Proof. Note that $\pi_{\underline{\sigma},\overline{\sigma}}(F) \geq \overline{\pi}(F)$. It suffices to show that for any $\varepsilon > 0$ there exists a self-financial strategy (η, θ) which super-replicates $F(S)$ in every scenario $\omega \in \Omega$ with initial position $(\varepsilon + \overline{\pi}(F), 0)$. Consider the Binomial model with volatility $\overline{\sigma}$. Let $\overline{\Omega} = \{-1,1\}^N$ be the path space with canonical process $\overline{X}_n(\overline{\omega}) \coloneqq \overline{x}_n$ for $\overline{\omega} = (\overline{x}_1, \dots, \overline{x}_N) \in \overline{\Omega}$ and the stock price evolution $\overline{S}_0 = s_0$ and $\overline{S}_n = \overline{S}_{n-1}\exp(\overline{\sigma}\overline{X}_n)$, $n = 1, \dots, N$. Clearly, the canonical filtration

$$\bar{\mathcal{F}}_n \coloneqq \sigma\left(\bar{X}_1, \dots, \bar{X}_n\right),\ n = 0, \dots, N \tag{10}$$

coincides with the one generated by $\overline{S} = \left(\overline{S}_n\right)_{n=0,\dots,N}$. By the definition of $\overline{\pi}(F)$ there is $\left(\overline{\mathcal{F}}_n\right)_{n=0,\dots,N}$ -predictable process $\left(\overline{\eta}, \overline{\theta}\right)$ such that with $\overline{\theta}_0 = 0$ we have

$$\vartheta_0\left(\varepsilon + \bar{\pi}(F) - \bar{\eta}_1, \bar{\theta}_0 - \bar{\theta}_1\right) \geq 0 \tag{11}$$

$$\vartheta_n\left(\bar{\eta}_n - \bar{\eta}_{n+1}, \bar{\theta}_n - \bar{\theta}_{n+1}\right) \geq 0,\ \ n = 1, \dots, N-1 \tag{12}$$

$$\vartheta_N\left(\bar{\eta}_N - F\left(\bar{S}\right), \bar{\theta}_N\right) \geq 0 \tag{13}$$

everywhere on $\overline{\Omega}$. In view of inequalities (2) for any $\omega \in \Omega$ and $n = 1, \dots, N$ there are unique weights $w_n^{(+1)}(\omega)$, $w_n^{(-1)}(\omega) \geq 0$ with $w_n^{(+1)}(\omega) + w_n^{(-1)}(\omega) = 1$ such that

$$e^{X_n(\omega)} = w_n^{(+1)}(\omega)e^{\bar{\sigma}} + w_n^{(-1)}(\omega)e^{-\bar{\sigma}}. \tag{14}$$

Observe that for weights

$$w_n^{\bar{\omega}^n}(\omega) := \prod_{m=1}^{n} w_n^{(\bar{\omega}_m^n)}(\omega),\ \bar{\omega}^n = \left(\bar{\omega}_1^n, \ldots, \bar{\omega}_n^n\right) \in \{-1,1\}^n \tag{15}$$

we have

$$\sum_{\bar{\omega}^n \in \{-1,1\}^n} w_n^{\bar{\omega}^n}(\omega) = \prod_{m=1}^{n}\left(w_m^{(+1)}(\omega) + w_m^{(-1)}(\omega)\right) = 1 \tag{16}$$

for $n = 1, \ldots, N$ and

$$w_n^{\bar{\omega}^n}(\omega) = w_n^{\bar{\omega}^n}(\omega) \prod_{m=n+1}^{N}\left(w_m^{(+1)}(\omega) + w_m^{(-1)}(\omega)\right) = \sum_{\bar{\omega}^{N-n} \in \{-1,1\}^{N-n}} w_N^{\left(\bar{\omega}^n, \bar{\omega}^{N-n}\right)}(\omega). \tag{17}$$

By using (3), (14), (17) and the adaptedness of $\bar{S}$ we get the following representation

$$\begin{aligned} S_n(\omega) &= s_0 \prod_{m=1}^{n}\left(w_m^{(+1)}(\omega) e^{\bar{\sigma}} + w_m^{(-1)}(\omega) e^{-\bar{\sigma}}\right) \\ &= \sum_{\bar{\omega}^n \in \{-1,1\}^n} \bar{S}_n\left(\bar{\omega}^n, 1, \ldots, 1\right) w_n^{\bar{\omega}^n}(\omega) = \sum_{\bar{\omega} \in \bar{\Omega}} \bar{S}_n(\bar{\omega}) w_N^{\bar{\omega}}(\omega) \end{aligned} \tag{18}$$

for any $n = 1, \ldots, N$, $\omega \in \Omega$. Now consider the pair (η, θ) of predictable processes η_n, θ_n:

$$\eta_n(\omega) := \begin{cases} \varepsilon + \bar{\pi}(F), & n = 0 \\ \bar{\eta}_1, & n = 1 \\ \sum_{\bar{\omega}^{n-1} \in \{-1,1\}^{n-1}} \bar{\eta}_n\left(\bar{\omega}^{n-1}, 1, \ldots, 1\right) w_{n-1}^{\bar{\omega}^{n-1}}(\omega), & n = 2, \ldots, N \end{cases} \tag{19}$$

$$\theta_n(\omega) := \begin{cases} 0, & n = 0 \\ \bar{\theta}_1, & n = 1 \\ \dfrac{1}{S_{n-1}(\omega)} \sum_{\bar{\omega}^{n-1} \in \{-1,1\}^{n-1}} \bar{\theta}_n \bar{S}_{n-1}\left(\bar{\omega}^{n-1}, 1, \ldots, 1\right) w_{n-1}^{\bar{\omega}^{n-1}}(\omega), & n = 2, \ldots, N. \end{cases} \tag{20}$$

First we prove that the strategy (η, θ) is self-financing. In view of (11), (7), (16), (19) and (20) we have

$$\begin{aligned} 0 &\le \sum_{\bar{\omega} \in \bar{\Omega}} w_N^{\bar{\omega}}(\omega) \vartheta_0\left(\varepsilon + \bar{\pi}(F) - \bar{\eta}_1(\bar{\omega}), \bar{\theta}_0(\bar{\omega}) - \bar{\theta}_1(\bar{\omega})\right) \\ &= \vartheta_0\left(\varepsilon + \bar{\pi}(F) - \bar{\eta}_1(\bar{\omega}), \bar{\theta}_0(\bar{\omega}) - \bar{\theta}_1(\bar{\omega})\right) \\ &= \vartheta_0\left(\varepsilon + \bar{\pi}(F) - \eta_1(\omega), \theta_0(\omega) - \theta_1(\omega)\right). \end{aligned} \tag{21}$$

Observe that

$$\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{\eta}_n(\bar{\omega}) = \sum_{\bar{\omega}^{n-1}\in\{-1,1\}^{n-1}} w_{n-1}^{\bar{\omega}^{n-1}}(\omega)\bar{\eta}_n\left(\bar{\omega}^{n-1},1,\ldots,1\right) = \eta_n(\omega) \tag{22}$$

for any $n = 1, \ldots, N, \omega \in \Omega$. Since $\bar{\theta}$ is predictable with respect to the filtration $(\mathcal{F}_n)_{n=0,\ldots,N}$ and $\bar{S}$ is $(\mathcal{F}_n)_{n=0,\ldots,N}$-adapted we have

$$\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{\theta}_{n+1}(\bar{\omega})\bar{S}_n(\bar{\omega}) = \sum_{\bar{\omega}^{n}\in\{-1,1\}^{n}} w_n^{\bar{\omega}^n}(\omega)(\bar{\theta}_{n+1}\bar{S}_n)\left(\bar{\omega}^{n},1,\ldots,1\right) = \theta_{n+1}(\omega)S_n(\omega) \tag{23}$$

for any $n = 1, \ldots, N-1, \omega \in \Omega$ and

$$\begin{aligned}
&\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{\theta}_n(\bar{\omega})\bar{S}_n(\bar{\omega}) = \sum_{\bar{\omega}^{n}\in\{-1,1\}^{n}} w_n^{\bar{\omega}^n}(\omega)(\bar{\theta}_n\bar{S}_n)\left(\bar{\omega}^{n},1,\ldots,1\right) \\
&= \sum_{\bar{\omega}^{n-1}\in\{-1,1\}^{n-1}} \sum_{\bar{x}\in\{-1,1\}} w_{n-1}^{\bar{\omega}^{n-1}}(\omega) w_n^{\bar{x}}(\omega)\left(\bar{\theta}_n\bar{S}_n\right)\left(\bar{\omega}^{n-1},\bar{x},\ldots,1\right) \\
&= \sum_{\bar{\omega}^{n-1}\in\{-1,1\}^{n-1}} w_{n-1}^{\bar{\omega}^{n-1}}(\omega)\left(\bar{\theta}_n\bar{S}_{n-1}\right)\left(\bar{\omega}^{n-1},1,\ldots,1\right)\left[w_n^{(+1)}(\omega)e^{\bar{\sigma}} + w_n^{(-1)}(\omega)e^{-\bar{\sigma}}\right] \\
&= \theta_n(\omega)S_{n-1}(\omega)e^{X_n(\omega)} = \theta_n(\omega)S_n(\omega)
\end{aligned} \tag{24}$$

for any $n = 2, \ldots, N-1, \omega \in \Omega$. Now, using (12) and (7) we obtain

$$\begin{aligned}
&0 \le \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\vartheta_n\left(\bar{\eta}_n(\bar{\omega}) - \bar{\eta}_{n+1}(\bar{\omega}), \bar{\theta}_n(\bar{\omega}) - \bar{\theta}_{n+1}(\bar{\omega})\right) \\
&= \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\eta}_n(\bar{\omega}) - \bar{\eta}_{n+1}(\bar{\omega})\right) + \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_n(\bar{\omega}) - \bar{\theta}_{n+1}(\bar{\omega})\right)\bar{S}_n(\bar{\omega}) \\
&-\mu\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_n(\bar{\omega}) - \bar{\theta}_{n+1}(\bar{\omega})\right)^+ \bar{S}_n(\bar{\omega}) - \lambda\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_n(\bar{\omega}) - \bar{\theta}_{n+1}(\bar{\omega})\right)^- \bar{S}_n(\bar{\omega})
\end{aligned} \tag{25}$$

for any $n = 1, \ldots, N-1, \omega \in \Omega$, which in conjunction with (22), (23), (24) and the convexity of the positive and the negative part entails the inequality

$$0 \le \vartheta_n\left(\eta_n(\omega) - \eta_{n+1}(\omega), \theta_n(\omega) - \theta_{n+1}(\omega)\right) \tag{26}$$

for any $n = 1, \ldots, N-1, \omega \in \Omega$. It remains to prove the super-replication condition

$$\vartheta_N\left(\eta_N(\omega) - F(S), \theta_N(\omega)\right) \ge 0 \tag{27}$$

for $\omega \in \Omega$. By definition of ϑ_N and (13) we have

$$\begin{aligned}
&0 \le \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\vartheta_N\left(\bar{\eta}_N(\bar{\omega}) - F\left(\bar{S}(\bar{\omega})\right), \bar{\theta}_N(\bar{\omega})\right) \\
&= \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\eta}_N(\bar{\omega}) - F\left(\bar{S}(\bar{\omega})\right)\right) + \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{\theta}_N(\bar{\omega})\bar{S}_N(\bar{\omega}) \\
&-\mu\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_N(\bar{\omega})\right)^+ \bar{S}_N(\bar{\omega}) - \lambda\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_N(\bar{\omega})\right)^- \bar{S}_N(\bar{\omega}).
\end{aligned} \tag{28}$$

Note that

$$\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega) F\left(\bar{S}(\bar{\omega})\right) \geq F\left(\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{S}_0(\bar{\omega}),\ldots,\sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\bar{S}_N(\bar{\omega})\right)$$
$$= F\left(S_0(\omega),\ldots,S_N(\omega)\right) = F\left(S(\omega)\right) \qquad (29)$$

due to the convexity of $F: \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ and (18). Similarly,

$$\mu \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_N(\bar{\omega})\right)^+ \bar{S}_N(\bar{\omega}) \geq \mu\theta_N^+(\omega) S_N(\omega) \qquad (30)$$

$$\lambda \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\left(\bar{\theta}_N(\bar{\omega})\right)^- \bar{S}_N(\bar{\omega}) \geq \lambda\theta_N^-(\omega) S_N(\omega). \qquad (31)$$

As a consequence,

$$0 \leq \sum_{\bar{\omega}\in\bar{\Omega}} w_N^{\bar{\omega}}(\omega)\vartheta_N\left(\bar{\eta}_N(\bar{\omega}) - F\left(\bar{S}(\bar{\omega})\right), \bar{\theta}_N(\bar{\omega})\right)$$
$$\leq \vartheta_N\left(\eta_N(\omega) - F\left(S(\omega)\right), \theta_N(\omega)\right). \qquad (32)$$

This shows that the self-financing strategy (η, θ) super-replicates $F(S)$ with $\eta_0 = \varepsilon + \overline{\pi}(F)$, which completes our proof.

4. Consequences and further generalizations

In our setup the set of all possible stock price evolutions which respect the specified volatility bounds is uncountable, but in view of Theorem 3, from Theorem 4.2 in [Roux, Tokarz and Zastawniak 2008], we obtain

Corollary 4 *The super-replication price of a European option with convex payoff $F(S)$ is given by*

$$\pi_{\underline{\sigma},\bar{\sigma}}(F) = \max_{(\mathsf{P},S)\in\mathsf{P}} \mathsf{E}_{\mathsf{P}}\left(F(S)\right) = \max_{x\in\mathsf{R}} Z_0^x - (1+\lambda)S_0,$$

where $\mathcal{P}$ denotes the set of pairs $(\mathbb{P}, \hat{S})$ such that $\mathbb{P}$ is probability measure on $\overline{\Omega}$ and $\hat{S}$ is a martingale under $\mathbb{P}$ satisfying

$$S_n(1-\mu) \leq S_n \leq S_n(1+\lambda)$$

for any $n = 0, \ldots, N$, Z_0 is the polyhedral proper convex function constructed as follows:

- *we put*

$$Z_N^x = \tilde{Z}_N^x = \begin{cases} F(S) + x & \text{if } x \in \left[S_N(1-\mu), S_N(1+\lambda)\right] \\ -\infty & \text{if } x \notin \left[S_N(1-\mu), S_N(1+\lambda)\right] \end{cases};$$

- *for any $n = 0, \ldots, N-1$ we take*

$$Z_n\left(\overline{\omega}^n\right)=cap\ \left\{Z_{n+1}\left(\overline{\omega}^{n+1}\right):\overline{\omega}^{n+1}\in\left\{\left(\overline{\omega}^n,1\right),\left(\overline{\omega}^n,-1\right)\right\}\right\}$$ [1]

and

$$Z_n^x=\begin{cases}Z_n^x & if\ x\in\left[S_n\left(1-\mu\right),S_n\left(1+\lambda\right)\right]\\ -\infty & if\ x\notin\left[S_n\left(1-\mu\right),S_n\left(1+\lambda\right)\right].\end{cases}$$

This method allows us to price European option in a algorithmic way as was studied in [Roux, Tokarz and Zastawniak 2008].

In the following paper proportional transaction costs have been considered . It seems we can extend the result to convex transaction costs. Further extension will go to consider multi-asset case and generalized the result in this direction.

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[1] The *concave cap* $cap\{f_1,\dots,f_n\}$ of functions $f_1,\dots,f_n:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is defined as the smallest concave function h such that $h\geq\max\{f_1,\dots,f_n\}$.

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